Date ________

Differentiability of a function:

Let $f(x)$ be a real function and a be any real number. we define

① Right hand derivative of $f(x)$ as

$$\boxed{Rf'(a)} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Right hand derivative of $f(x)$ at $x=a$

② Left hand derivative of $f(x)$ as

$$\boxed{Lf'(a)} = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$$

left hand derivative of $f(x)$ at $x=a$

If these two limits exist and are equal, then the function is said to be differentiable at $x=a$

So, a function is differentiable only if the right hand derivative and left hand derivative both exists and are equal.

e.g

① Show that the function $f(x)=x^2$ is differentiable at $x=1$ and find $f'(1)$.

Let us evaluate Right hand derivative.

$$Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

here $a=1$ in above definition

$$= \lim_{h\to 0} \frac{(1+h)^2-(1)^2}{h}$$

$$= \lim_{h\to 0} \frac{1+h^2+2h-1}{h} = \lim_{h\to 0}(h+2) = \boxed{2}$$

$$Lf'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0} \frac{(1-h)^2-(1)^2}{-h}$$

$$= \lim_{h\to 0} \frac{1^2+h^2-2h-1}{-h} = \lim_{h\to 0}(-h+2)$$

$$= \boxed{2}$$

So $Rf'(1) = Lf'(1) \Rightarrow f(x)$ is differentiable at $x=1$ and $f'(1)=2$

Date

② Show that $f(x) = [x]$ is not differentiable at $x = 1$

Solution:-

$$Rf'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{[1+h] - [1]}{h}$$

$$= \lim_{h\to 0} \frac{1-1}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$ {as h is a small value tending to zero}

$$Lf'(1) = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{[1-h] - [1]}{-h}$$

$$= \lim_{h\to 0} \frac{0-1}{-h} = \lim_{h\to 0} \frac{1}{h} = \infty$$ {$[1-h] = 0$ $(1-h)$ means less than 1.}

So

$Rf'(1) \neq Lf'(1) \Rightarrow f(x)$ is not differentiable at $x = 1$.

③ Do yourself

Show that

$$f(x) = \begin{cases} (1+x) & ; \text{ if } x \le 2 \\ (5-x) & ; \text{ if } x > 2 \end{cases}$$

is not differentiable at $x = 2$

Hint: { evaluate $Rf'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$

$Lf'(2) = \lim_{h\to 0} \frac{f(2-h) - f(2)}{-h}$

Take the function accordingly for $x \geq 2$ & $x < 2$ as given in the ~~above~~ ~~qu~~ question.

Date

Note :- Every differentiable function is continuous.
But every continuous function need not be differentiable.

Let us check for the function $f(x) = |x-2|$ two things
(1) continuity at $x=2$
(2) differentiability at $x=2$

(1) Continuity

Consider $f(2) = |2-2| = 0$

Right hand limit $x=2$

$$\lim_{x\to 2+} f(x) = \lim_{h\to 0} |2+h-2| = \lim_{h\to 0} |h| = 0$$

Left hand limit

$$\lim_{x\to 2-} f(x) = \lim_{h\to 0} |2-h-2| = \lim_{h\to 0} |-h| = 0$$

So limit of the function exists and is equal to the value of the function at $x=0$
$\Rightarrow$ $f(x)$ is continuous at $x=2$.

(2) Differentiability

$$Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{|2+h-2|-0}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = 1$$

$$Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0} \frac{|2-h-2|-0}{-h}$$

$$= \lim_{h\to 0} \frac{h}{-h} = -1$$

Since $Rf'(2) \neq Lf'(2) \Rightarrow f(x)$ is not differentiable at $x=2$.

Thus the function is continuous but is not differentiable at $x=2$.

Date

Practice questions:

1. Give an example to show that the function is continuous at $x=0$ but not differentiable at $x=0$

2. Show that the function $f(x)=|x-5|$ is not differentiable at $x=5$.

3. Find the values of a and b so that the function

$$f(x)=\begin{cases} x^2+3x+a & ; \quad x\leq 1 \\ (bx+2) & ; \quad x>1 \end{cases}$$

is differentiable at each $x \in R$.